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Exact statistics of complex zeros for Gaussian random polynomials with real coefficients

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Abstract. k-point correlations of *complex zeros* for Gaussian ensembles of *random polynomials* of order N with *real* coefficients (GRPRC) are calculated exactly, following an approach of Hannay [5] for the case of Gaussian random polynomials with complex coefficients (GRPCC). It is shown that in the thermodynamic limit $N \to \infty$ of Gaussian random holomorphic functions all the statistics converge to their GRPCC counterparts as one moves off the real axis, while close to the real axis the two cases are essentially different. Special emphasis is given to one- and two-point correlation functions in various regimes.

The problem of statistics of zeros of random polynomials of order N, and of random holomorphic functions as $N \to \infty$ in general, arises in various contexts in quantum chaos [2, 3]. The motivation for this work was the problem of statistics of zeros of coherent state (Husimi) or Bargmann [4] representation of eigenstates of chaotic systems [6, 8]. It has been conjectured [6] that zeros of Bargmann or Husimi representation of an eigenfunction of 1-dim classically chaotic system should be uniformly and randomly scattered over the classically chaotic region of phase space. A Bargmann representation of an eigenstate is an entire analytic function in a complex phase space variable $z = q + \mathrm{i}p$, sometimes it is even a polynomial of a finite order, like for example in the case of spin systems where the phase space manifold is a sphere parametrized by (θ, ϕ) and $z = \cot(\theta/2)\exp(\mathrm{i}\phi)$ is a stereographic projection. The coefficients of a power series of such entire functions or polynomials are just the coefficients of an expansion of the chaotic eigenstate in a complete set of (say harmonic) wavefunctions. Applying the random matrix theory one argues that these coefficients should be uncorrelated (real/complex in the presence/absence of anti-unitary symmetry) pseudorandom Gaussian variables. Thus one can introduce the statistical ensembles of random polynomials of order N (or random analytic functions in the limit $N \to \infty$) and argue that statistical properties of their zeros can be used as a model to describe statistical properties of zeros of a Bargman representation of chaotic eigenstates of real systems.

Recently, Hannay [5] has calculated general k-point correlation functions of zeros of a random spin state in a coherent state representation which is described by the random

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polynomial with uncorrelated complex Gaussian coefficients, and solved the problem of statistics of zeros of GRPCC—Gaussian random polynomials with *complex* coefficients in general. It has been demonstrated numerically [7, 9] that his results on GRPCC provide a universal description of the statistics of zeros of Bargmann or Husimi representation of chaotic eigenstates for systems without an anti-unitary symmetry. Here we adopt this approach and solve the general problem of statistics of zeros z_k of GRPRC—random polynomials $f(z)$ of order N

$$f(z) = \sum_{n=0}^{N} a_n z^n = a_N \prod_{j=1}^{N} (z - z_j) \tag{1}$$

with *real* (Gaussian) coefficients a_n. We argue that the results obtained may be used to describe statistics of zeros of eigenstates of 1-dim (and quantum Poincaré sections [8] and other reductions [10, 11] of 2-dim) chaotic systems in Bargmann representation with *time reversal invariance* (or any other anti-unitary symmetry†) to the same extent as Gaussian orthogonal ensembles of random matrices can be used to describe the Hamiltonian and the typical observables.

In the literature one may find several results on the distribution of complex zeros of random polynomials with either complex [1] or real [12] Gaussian coefficients (see also [3] and references therein). The formula for the one-point function given below (19) (in the special case where the variances of all coefficients are equal) is equivalent to theorem 1.1 of Shepp and Vanderbei [12].

Take a k-tuple of complex numbers $\boldsymbol{z} = (z_1, \ldots, z_k)$. Since a_n are real Gaussian random variables (which in general need not be uncorrelated!), their real linear combinations

$$\begin{aligned} &f_j^r = \mathrm{Re}\, f(z_j) \qquad f_j^i = \mathrm{Im}\, f(z_j) \\ &f_j'^r = \mathrm{Re}\,\frac{\mathrm{d}}{\mathrm{d}z} f(z_j) \qquad f_j'^i = \mathrm{Im}\,\frac{\mathrm{d}}{\mathrm{d}z} f(z_j) \qquad j = 1, \ldots, k \end{aligned} \tag{2}$$

are also real Gaussian random variables with a joint distribution

$$P(\boldsymbol{f}^r, \boldsymbol{f}^i, \boldsymbol{f}'^r, \boldsymbol{f}'^i) = (\det 2\pi \tilde{\mathbf{M}})^{-1/2} \exp\big(-\tfrac{1}{2}(\boldsymbol{f}^r, \boldsymbol{f}^i, \boldsymbol{f}'^r, \boldsymbol{f}'^i) \cdot \tilde{\mathbf{M}}^{-1} (\boldsymbol{f}^r, \boldsymbol{f}^i, \boldsymbol{f}'^r, \boldsymbol{f}'^i)\big). \tag{3}$$

$\tilde{\mathbf{M}}$ is a $4k \times 4k$ real symmetric positive covariance matrix

$$\tilde{\mathbf{M}} = \begin{pmatrix} \langle f_j^r f_l^r \rangle & \langle f_j^r f_l^i \rangle & \langle f_j^r f_l'^r \rangle & \langle f_j^r f_l'^i \rangle \\ \langle f_j^i f_l^r \rangle & \langle f_j^i f_l^i \rangle & \langle f_j^i f_l'^r \rangle & \langle f_j^i f_l'^i \rangle \\ \langle f_j'^r f_l^r \rangle & \langle f_j'^r f_l^i \rangle & \langle f_j'^r f_l'^r \rangle & \langle f_j'^r f_l'^i \rangle \\ \langle f_j'^i f_l^r \rangle & \langle f_j'^i f_l^i \rangle & \langle f_j'^i f_l'^r \rangle & \langle f_j'^i f_l'^i \rangle \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \tilde{\mathbf{B}}^{\mathrm{T}} & \tilde{\mathbf{C}} \end{pmatrix} \tag{4}$$

where $\langle\ \rangle$ denotes the Gaussian ensemble averages which can be calculated using (1), (2) in terms of input data $\langle a_n a_m \rangle$. One can write the k-point correlation function $\rho_k(\boldsymbol{z})$ in the following form

$$\rho_k(\boldsymbol{z}) = \int P(\mathbf{0}, \mathbf{0}, \boldsymbol{f}'^r, \boldsymbol{f}'^i) \prod_{j=1}^{k} [(f_j'^r)^2 + (f_j'^i)^2]\, \mathrm{d}f_j'^r\, \mathrm{d}f_j'^i \tag{5}$$

† For a general anti-unitary symmetry, the coefficients of the random polynomials (1) are of the form $a_n = r_n \mathrm{e}^{\mathrm{i}\theta_n}$ where r_n are real Gaussian random variables and θ_n are fixed (nonrandom) phases (which determine the symmetry curve in complex z-plane, such as in figure 6 of [3]). Then one may use the same general approach described below, equations (2)–(13).

where the factors $(f_j'^r)^2 + (f_j'^i)^2$ are just the Jacobians of transformations from the pairs of real variables (f_j^r, f_j^i) to complex variables—zeros z_j. The integral can be written in terms of derivatives of a generating function $Z_k(\boldsymbol{u}, \boldsymbol{v})$

$$\rho_k(\boldsymbol{z}) = (-1)^k \prod_{j=1}^{k} (\partial_{u_j}^2 + \partial_{v_j}^2) Z_k(\boldsymbol{u}, \boldsymbol{v})|_{\boldsymbol{u}=\boldsymbol{v}=0} \tag{6}$$

which is an ordinary Gaussian integral and can be explicitly calculated

$$\begin{aligned} Z_k(\boldsymbol{u}, \boldsymbol{v}) &= (\det 2\pi \tilde{\mathbf{M}})^{-1/2} \int \exp\big(-\tfrac{1}{2}(\boldsymbol{f}'^r, \boldsymbol{f}'^i) \cdot \tilde{\mathbf{L}}(\boldsymbol{f}'^r, \boldsymbol{f}'^i) + \mathrm{i}\boldsymbol{f}'^r \cdot \boldsymbol{u} + \mathrm{i}\boldsymbol{f}'^i \cdot \boldsymbol{v}\big) \\ &\quad \times \prod_{j=1}^{k} \mathrm{d}f_j'^r \, \mathrm{d}f_j'^i \\ &= (\det 2\pi \tilde{\mathbf{A}})^{-1/2} \exp\big(-\tfrac{1}{2}(\boldsymbol{u}, \boldsymbol{v}) \cdot \tilde{\mathbf{L}}(\boldsymbol{u}, \boldsymbol{v})\big) \end{aligned} \tag{7}$$

where $\tilde{\mathbf{L}} = \tilde{\mathbf{C}} - \tilde{\mathbf{B}}^{\mathrm{T}} \tilde{\mathbf{A}}^{-1} \tilde{\mathbf{B}}$ is a lower right block of $\tilde{\mathbf{M}}^{-1}$ and we have used an identity [5] $\det \tilde{\mathbf{L}} / \det \tilde{\mathbf{M}} = 1 / \det \tilde{\mathbf{A}}$. At this point it is convenient to switch on the equivalent complex variables $\boldsymbol{f} = \boldsymbol{f}^r + \mathrm{i}\boldsymbol{f}^i$, $\boldsymbol{f}' = \boldsymbol{f}'^r + \mathrm{i}\boldsymbol{f}'^i$, $\boldsymbol{w} = \boldsymbol{u} + \mathrm{i}\boldsymbol{v}$ and their complex conjugates. Then one can write equations (6), (7) as

$$\begin{aligned} \rho_k(\boldsymbol{z}) &= \frac{(-1)^k 2^k}{(\det 2\pi \mathbf{A})^{1/2}} \prod_{j=1}^{k} \partial_{w_j} \partial_{w_j^*} \exp\big(-\tfrac{1}{2}(\boldsymbol{w}^*, \boldsymbol{w}) \cdot \mathbf{L}(\boldsymbol{w}, \boldsymbol{w}^*)\big)|_{\boldsymbol{w}=0} \\ &= (\det 2\pi \mathbf{A})^{-1/2} \prod_{j=1}^{k} \partial_{w_j} \partial_{w_j^*} ((\boldsymbol{w}^*, \boldsymbol{w}) \cdot \mathbf{L}(\boldsymbol{w}, \boldsymbol{w}^*))^k |_{\boldsymbol{w}=0} \end{aligned} \tag{8}$$

where all the $2k \times 2k$ real matrices should be transformed by the rule

$$\mathbf{X} = \mathbf{U}^\dagger \tilde{\mathbf{X}} \mathbf{U} \qquad \mathbf{U} = \tfrac{1}{2} \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathrm{i}\mathbf{1} & -\mathrm{i}\mathbf{1} \end{pmatrix}$$

giving $\mathbf{L} = \mathbf{C} - \mathbf{B}^\dagger \mathbf{A}^{-1} \mathbf{B}$ with

$$\mathbf{A} = \begin{pmatrix} \langle f_j f_k^* \rangle & \langle f_j f_k \rangle \\ \langle f_j^* f_k^* \rangle & \langle f_j^* f_k \rangle \end{pmatrix} = \mathbf{A}^\dagger \tag{9}$$

$$\mathbf{B} = \begin{pmatrix} \langle f_j f_k'^* \rangle & \langle f_j f_k' \rangle \\ \langle f_j^* f_k'^* \rangle & \langle f_j^* f_k' \rangle \end{pmatrix} \tag{10}$$

$$\mathbf{C} = \begin{pmatrix} \langle f_j' f_k'^* \rangle & \langle f_j' f_k' \rangle \\ \langle f_j'^* f_k'^* \rangle & \langle f_j'^* f_k' \rangle \end{pmatrix} = \mathbf{C}^\dagger. \tag{11}$$

Applying a little combinatorics on (8) we finally obtain the general result

$$\rho_k(\boldsymbol{z}) = \frac{\operatorname{sper}(\mathbf{C} - \mathbf{B}^\dagger \mathbf{A}^{-1} \mathbf{B})}{\sqrt{\det 2\pi \mathbf{A}}} \tag{12}$$

where we introduce the *semi-permanent* of a $2k \times 2k$ matrix

$$\operatorname{sper} \mathbf{L} = \sum_{\substack{j_1 < \cdots < j_k \\ l_1 < \cdots < l_k}}^{j_m \neq l_n} \sum_{p \in S_k} \prod_{r=1}^{k} L_{j_r + k, l_{p(r)}}. \tag{13}$$

The first sum runs over $(2k)!/(k!)^2$ ordered combinations of k out of $2k$ indices j_m and their complements l_n while the second sum runs over $k!$ permutations p of the symmetric group S_k. The sum of indices $j_r + k$ should be taken modulo $2k$.

So far we have not assumed anything about the correlations between the coefficients a_n except the Gaussian nature of the joint distribution of coefficients a_n. Now we shall assume that Gaussian coefficients a_n are uncorrelated and define the polynomial $g(s)$ with positive coefficients b_n—the variances of a_n

$$\langle a_n a_m \rangle = b_n \delta_{nm} \qquad b_n > 0 \tag{14}$$

$$g(s) = \sum_{n=0}^{N} b_n s^n. \tag{15}$$

The matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ can be easily expressed solely in terms of a polynomial g and its derivatives g', g''

$$\mathbf{A}_{jl}(\boldsymbol{z}) = g(z_j z_l^*) \tag{16}$$

$$\mathbf{B}_{jl}(\boldsymbol{z}) = \partial_{z_l^*} \mathbf{A}_{jl}(\boldsymbol{z}) = z_j g'(z_j z_l^*) \tag{17}$$

$$\mathbf{C}_{jl}(\boldsymbol{z}) = \partial_{z_j} \partial_{z_l^*} \mathbf{A}_{jl}(\boldsymbol{z}) = g'(z_j z_l^*) + z_j z_l^* g''(z_j z_l^*) \tag{18}$$

where we let indices j, l to run from 1 through $2k$ and put $z_{k+j} := z_j^*$. Note that the time-reversal symmetry—the symmetry of zeros with respect to the reflection over the real axis—is present also in the k-point correlation functions, namely

$$\rho_k(z_1, \ldots, z_j, \ldots, z_k) = \rho_k(z_1, \ldots, z_j^*, \ldots, z_k).$$

Without loss of generality one may assume that all points z_j lie on the upper complex halfplane, $\mathrm{Im}\, z_j > 0$. Otherwise one gets long-range correlations in cases where one of the points z_j comes close to the mirror image of one of the other points z_l^*.

In general, only the one-point function $\rho_1(z)$—the density of zeros—is simple enough to be written out

$$\rho_1(z) = \frac{g_0' + |z|^2 g_0''}{\pi (g_0^2 - g_+ g_-)^{1/2}} + \frac{(z^2 g_- g_+' + z^{*2} g_+ g_-') g_0' - |z|^2 (g_+' g_-' + g_0'^2) g_0}{\pi (g_0^2 - g_+ g_-)^{3/2}} \tag{19}$$

where $g_0 \equiv g(|z|^2)$, $g_+ \equiv g(z^2)$, $g_- \equiv g(z^{*2})$. Writing $z = x + \mathrm{i}y$ and carefully expanding for small y one finds

$$\rho_1(z) = h(x^2)|y| + \mathcal{O}(y^3) \qquad y \neq 0$$

$$h(s) = (2\pi)^{-1} (g g' - s g'^2 + s g g'')^{-3/2} (2 g_{012} + 2(2 g_{013} - g_{112} - g_{022}) s + (3 g_{122} - 4 g_{113} + g_{014}) s^2 + (g_{024} - g_{114} - g_{033} + 2 g_{123} - g_{222}) s^3) \tag{20}$$

where $g \equiv g(s)$, $g_{nml} \equiv g^{(n)}(s) g^{(m)}(s) g^{(l)}(s)$. So quite generally, the density of zeros decreases linearly as we approach the real axis. To evaluate the density of zeros on a real axis $y = 0$ one should use a different approach described in [3]. In another asymptotical regime $|z| \to \infty$, only the highest power terms of g contribute, and one finds

$$\rho_1(z) = \frac{2 b_{N-2}}{\sqrt{b_N b_{N-1}}} \frac{\mathrm{Im}\, z}{|z|^6} \left(1 + \mathcal{O}\left(\frac{1}{|z|^2}\right)\right). \tag{21}$$

So, the density of zeros vanishes asymptotically since the total number of zeros N is finite.

Now we shall study the thermodynamic limit $N \to \infty$. It is convenient to study *random holomorphic functions* which provide a uniform distribution of zeros in the complex plane. A unique choice (up to rescaling $s \to \lambda s$) is $b_n = 1/n!$ giving

$$g(s) = \exp(s). \tag{22}$$

Such random holomorphic functions naturally arise when one studies Bargmann representation of 1-dim chaotic eigenstates in the usual $(p, q) \in \Re^2$ phase space. We argue

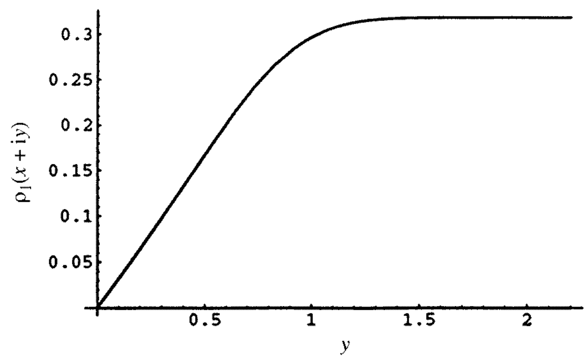

Figure 1. The density of zeros $\rho_1(x+\mathrm{i}y)$ in the thermodynamic limit $N\to\infty$ given by equation (26) as a function of the distance from the real axis.

that any other choice will only affect the density of zeros $\rho_1(z)$ while properly rescaled local statistics should be independent on the choice of $g(s)$ provided that variances of coefficients b_n *depend smoothly* on n.

Far enough away from the real axis $\operatorname{Im} z_j \gg 1$ one may neglect the off-diagonal $k\times k$ blocks of matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ since the ratios of the corresponding matrix elements become exponentially small $|\exp(z_j z_l^*)/\exp(z_j z_l)| = \exp(-2\operatorname{Im} z_j \operatorname{Im} z_l)$. Then using straightforward results

$$2^{-k}\,\mathrm{sper}\begin{pmatrix}\mathbf{L}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{L}_{11}^{\mathrm{T}}\end{pmatrix} = \mathrm{per}\,\mathbf{L}_{11} := \sum_{p\in S_k}\prod_{j=1}^{k} L_{j,p(j)} \tag{23}$$

$$\det\begin{pmatrix}\mathbf{A}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_{11}^{\mathrm{T}}\end{pmatrix} = (\det\mathbf{A}_{11})^2 \tag{24}$$

where $(\ )_{11}$ denotes the upper-left $k\times k$ block of a $2k\times 2k$ matrix, one arrives at the result which is equivalent to the statistics of zeros of GRPCC [5]

$$\rho_k(\boldsymbol{z}) \to \rho_k^{\mathrm{GRPCC}}(\boldsymbol{z}) = \frac{\mathrm{per}(\mathbf{C}_{11} - \mathbf{B}_{11}^{\dagger}\mathbf{A}_{11}^{-1}\mathbf{B}_{11})}{\det \pi\mathbf{A}_{11}} \qquad \text{as } \operatorname{Im} z_j \to \infty. \tag{25}$$

To conclude we give some explicit results about one- and two-point functions. The density of zeros which is shown in figure 1 reads

$$\rho_1(x+\mathrm{i}y) = \frac{1-(4y^2+1)\exp(-4y^2)}{\pi(1-\exp(-4y^2))^{3/2}} \tag{26}$$

which is a constant $1/\pi$ provided that we are far enough away from the real axis. The excess of zeros due to the presence of real axis $\int_{-\infty}^{\infty}(1/\pi - \rho_1(x+\mathrm{i}y))\,\mathrm{d}y = 1/\pi$ is, on the other hand, just the linear density of real zeros on the real axis!

The two-point correlation function $\rho_2(z_1,z_2)$ is already too lengthy to be written out in general. The behaviour of a normalized two-point correlation function $\rho_2(z_1,z_2)/\rho_1(z_1)/\rho_1(z_2)$ as we approach the real axis is shown in figure 2, while far away $\operatorname{Im} z_1, \operatorname{Im} z_2 \gg 1$ it becomes isotropic and the result for GRPCC [5] applies

$$\begin{aligned}&\rho_2(z_1,z_2) \to \varphi(|z_1-z_2|^2)\\ &\varphi(s) = \frac{\exp(-2s)(\exp(s)-1-s)^2 + \exp(-s)(\exp(-s)-1+s)^2}{\pi^2(1-\exp(-s))^3}.\end{aligned} \tag{27}$$

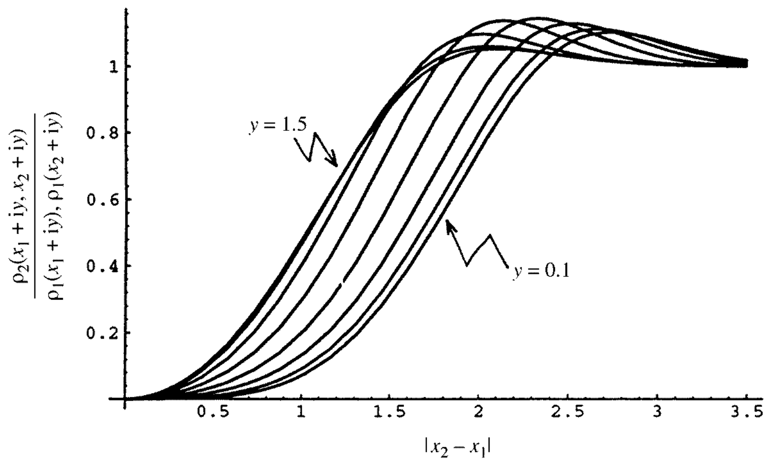

Figure 2. The normalized two-point correlation function $\rho_2(x_1+\mathrm{i}y, x_2+\mathrm{i}y)/\rho_1(x_1+\mathrm{i}y)/\rho_1(x_2+\mathrm{i}y)$ in the limit $N \to \infty$ between two points, $x_1 + \mathrm{i}y$ and $x_2 + \mathrm{i}y$, which have the same distance from the real axis y is shown as a function of $|x_2 - x_1|$ for different values of $y = 0.1, 0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5$. Note that all curves go to zero as $\propto y^2$ and that for $y \geqslant 1.5$ the two-point correlation function has practically converged to the isotropic asymptotic one.

In the asymptotic regime $\mathrm{Im}\, z_j \gg 1$ one can also calculate the *number variance* $\Sigma_2(r)$: the variance of the number of zeros $\mathcal{N}(r)$ inside a circle of radius r

$$\Sigma_2(r) = \langle \mathcal{N}^2(r)\rangle - \langle \mathcal{N}(r)\rangle^2. \tag{28}$$

It can be expressed in terms of a four-fold integral (over z_1, z_2) of a two-point correlation, which can be reduced using equation (27) to a single integral

$$\Sigma_2(r) = r^2(1-r^2) + 8\pi r^4 \int_0^1 \left(\arccos\sqrt{t} - \sqrt{t(1-t)}\right)\varphi(4r^2 t)\,\mathrm{d}t. \tag{29}$$

The number variance $\Sigma_2(r)$ starts as 'Poissonian' $\langle \mathcal{N}(r)\rangle = r^2$ for small r whereas for larger r it has a linear asymptotics (see figure 3)

$$\Sigma_2(r) = \sigma r + \mathcal{O}(1/r) \approx \sigma\sqrt{\langle \mathcal{N}(r)\rangle} \qquad \sigma = \frac{4}{\pi}\int_0^\infty s^2(1-\pi^2\varphi(s^2))\,\mathrm{d}s \approx 0.368\,47. \tag{30}$$

Note that this formula (29), (30) is valid also for GRPCC in general.

In the present paper the statistics of zeros of Gaussian random polynomials with real coefficients have been solved analytically (12) following an approach of Hannay for the case of complex coefficients. Several important special cases have been considered in detail: (i) the case of mutually uncorrelated coefficients, which corresponds to the Bargmann representation of chaotic eigenstates in the random matrix regime, has been studied and it has been shown that all k-point correlation functions converge to those of random polynomials with complex coefficients derived by Hannay as all points z_j, $j = 1, \ldots, k$ move away from the real axis $\mathrm{Im}\, z_j \gg 1$ (25); (ii) one-point functions—the density of zeros—have been written out in general (equations (19), (26) and figure 1) and linear decrease of density towards the symmetry line–real axis has been found (20) (iii) two-point functions close to the real axis have been explored numerically (figure 2) while the simple analytic formula (27), which holds far away from the real axis (and holds generally in the case of complex

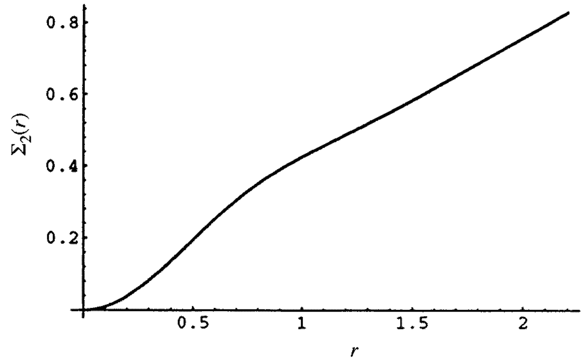

Figure 3. The number variance $\Sigma_2(r)$ in the asymptotical regime $N \to \infty$, $\mathrm{Im}\, z \gg 1$ is shown as a function of the radius r (29).

coefficients), has been used to derive a simple expression for the number variance of zeros inside a circle of a given radius (equations (29), (30) and figure 3).

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